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On the Order of Approximation of Continuous Functions by Positive Linear Operators of Finite Rank

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Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators from $C[0, 1]$ into $B(\Omega)$, where $B(\Omega)$ is the space of real bounded functions over $\Omega\subset[0, 1]$, $\operatorname{meas}(\Omega)>0$. Suppose that for each n the linear space $\{L_n f: f\in C[0, 1]\}$ has dimension $n+1$. It is shown that the quantity

$$n^2 \sum_{j=0}^{2} |L_n(t^j; x)-x^j|$$

does not tend to zero on a set of positive measure.

1. INTRODUCTION

It was already shown by P. P. Korovkin [1] that for the functions 1, $\cos x$, and $\sin x$ the order of approximation by positive linear polynomial operators $L_n(f; x)$ ($n\in\mathbb{N}$, $f\in C_{2\pi}$, $L_n f$ is a trigonometric polynomial of order n) can not be better than $1/n^2$ in the norm of uniform convergence. Ph. C. Curtis [2] and V. K. Dzyadyk [3] have generalized this result to the spaces $L^p[-\pi, \pi]$, $1\leqslant p<\infty$. One of the authors of this note [4] has extended the result of Korovkin to the setting of almost everywhere

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convergence. It was shown that almost everywhere at least one of the quantities

$$n^2[L_n(1;x)-1], \qquad n^2[L_n(\cos t;x)-\cos x],$$
$$n^2[L_n(\sin t;x)-\sin x] \tag{1.1}$$

does not tend to zero if the norms of the operators $\|L_n\|_{C_{2\pi}\to C_{2\pi}}$ are uniformly bounded. Furthermore, for convolution operators the same result holds for all points of $\mathbb{R}$. All these results were based on the Bernstein inequality for trigonometric polynomials of order n : $\|T_n'\|_{C_{2\pi}} \leqslant n\,\|T_n\|_{C_{2\pi}}$.

If instead of 1, cos x, and sin x, we take 1, x, and x^2, one can obtain the same results for algebraic polynomial operators.

V. S. Videnskу [5] has shown that neither the polynomial properties of operators nor the Bernstein inequality is necessary for the proof of these results; it is the dimensions of the operators' images that play the principal role.

Let $C[a,b]$ be the space of real continuous functions on $[a,b]$. A linear operator L mapping $C[a,b]$ into a linear space of finite dimension n is called an *operator of finite rank n*. Let L_n be a positive linear operator of rank $n+1$ defined on $C=C[0,1]$ and let $L_n(1;x)\equiv 1$. It was shown in [5] that

$$2\,\|L_n(t;x)-x\|_C+\|L_n(t^2;x)-x^2\|_C$$
$$\geqslant \|L_n((t-x)^2;x)\|_C \geqslant 1/4(n+1)^2. \tag{1.2}$$

Furthermore, it was shown in [5] that

$$\inf_{L_n\in\mathscr{L}_n} \|L_n((t-x)^2;x)\|_C \leqslant 1/4n^2, \tag{1.3}$$

where $\mathscr{L}_n$ is the class of positive linear operators of rank $n+1$ from $C[0,1]$ into itself satisfying the condition $L_n(1;x)\equiv 1$. A similar result holds in the trigonometric case.

In this paper, using basically the method of V. S. Videnskу, we give a generalization of these results and those of [4]. We shall show that for positive linear operators L_n with finite rank $n+1$, almost everywhere at least one of the quantities

$$n^2\,[L_n(t^j;x)-x^j], \qquad j=0,1,2,$$

does not tend to zero as $n\to\infty$, even if the sequence of norms $\|L_n\|$, $n\in\mathbb{N}$, is not bounded. Furthermore, inequalities (1.2) and (1.3) are true in the case of positive linear operators in spaces of functions defined on sets $\Omega\subset[0,1]$.

The same results hold for the quantities (1.1) in the trigonometric case.

2. Lemmas

Let $F \subset [0, 1]$ be an infinite closed set, and $m \in \mathbb{N}$. Designate by $0 = \xi_0 < \xi_1 < \cdots < \xi_{n_m} = 1$, where $n_m \leqslant m$, all the points of the set $\omega_m = \{\xi = k/m : [(k-1)/m, (k+1)/m) \cap [F \cup \{0\} \cup \{1\}] \neq \varnothing, 0 \leqslant k \leqslant m\}$, and put

$$F_m := \left\{ \bigcup_{k \in \nu_m} [\xi_{k-1}, \xi_k] \right\} \cup \{1\},$$

where

$$\nu_m = \{k : \xi_k - \xi_{k-1} = 1/m, [\xi_{k-1}, \xi_k) \cap F \neq \varnothing\}.$$

Note that $F_m \supset F$. Taking into account that the set $(0, 1) \backslash F$ is composed of disjoint open intervals, we get

$$\lim_{m \to \infty} \operatorname{meas}([0, 1] \backslash F_m) = \lim_{m \to \infty} \frac{m - n_m}{m} = \operatorname{meas}([0, 1] \backslash F).$$

Therefore

$$\lim_{m \to \infty} \operatorname{meas}(F_m) = \lim_{m \to \infty} \frac{n_m}{m} = \operatorname{meas}(F).$$

For each fixed natural number n choose $m = m(n)$ so that $n_m \leqslant n < n_{m+1}$; and hence

$$\lim_{n \to \infty} \frac{n}{m(n)} = \operatorname{meas}(F) \qquad (n, m = m(n) \in \mathbb{N}). \tag{2.1}$$

Now, designate by $\lambda_{nk}(x)$ $(n \in \mathbb{N};\ k = 0, 1, ..., n_m)$ the continuous functions on $[0, 1]$ such that $\lambda_{nk}(\xi_i) = \delta_{ki}$, where δ_{ki} is the Kronecker delta, $0 \leqslant i, k \leqslant n_m$, and $\lambda_{nk}(x)$ is linear on each segment $[\xi_{i-1}, \xi_i]$ $(i = 1, ..., n_m)$. Consider the sequence of positive linear operators on $C[0, 1]$ with ranks $n_m \leqslant n$ given by the equalities

$$\Lambda_n(f; x) = \sum_{k=0}^{n_m} f(\xi_k)\, \lambda_{nk}(x),$$

where,

$$n, m = m(n) \in \mathbb{N}, \qquad n_m \leqslant n < n_{m+1} \qquad (f \in C[0, 1], 0 \leqslant x \leqslant 1).$$

For every $f \in C[0, 1]$, the function $\Lambda_n(f; x)$ coincides with the function $f(x)$ at all points ξ_k, $k = 0, 1, ..., n_m$; it is also continuous on $[0, 1]$ and linear on each segment $[\xi_{k-1}, \xi_k]$ $(k = 1, ..., n_m)$.

In the following, $B(F)$, where $F \subset [0, 1]$, is the space of real bounded functions with the uniform norm over F: $\|f\|_{B(F)} = \sup_{x \in F} |f(x)|$.

LEMMA 1. *Let $F \subset [0, 1]$ be an infinite closed set. Then*

$$\lim_{n \to \infty} \{n^2 \|\Lambda_n((t-x)^2; x)\|_{B(F)}\} = [\text{meas}(F)]^2/4. \tag{2.2}$$

Proof. Because of $\Lambda_n(1; x) \equiv 1$ and $\Lambda_n(t; x) = x$, we have $\Lambda_n((t-x)^2; x) = \Lambda_n(t^2; x) - x^2$. Furthermore, for $x \in [\xi_{k-1}, \xi_k] \subset F_m$ this function is a second degree polynomial which vanishes at the points ξ_{k-1} and ξ_k. Therefore $0 \leqslant \Lambda_n((t-x)^2; x) = (x - \xi_{k-1})(\xi_k - x) \leqslant (\xi_k - \xi_{k-1})^2/4 \leqslant 1/4m^2$ for $\xi_{k-1} \leqslant x \leqslant \xi_k$; and hence

$$\|\Lambda_n((t-x)^2; x)\|_{B(F_m)} = 1/4m^2,$$

$$\|\Lambda_n((t-x)^2; x)\|_{B(F)} \geqslant (1 - \theta_m^2)/4m^2,$$

where $\theta_m = \text{meas}(F_m \backslash F)/\text{meas}(F_m)$; $\theta_m \to 0$ as $m \to \infty$, $\text{meas}(F) \neq 0$. As a consequence of the last result and of (2.1) we obtain

$$\lim_{n \to \infty} \{n^2 \|\Lambda_n((t-x)^2; x)\|_{B(F)}\}$$

$$= \lim_{n \to \infty} \{n^2 \|\Lambda_n((t-x)^2; x)\|_{B(F_m)}\} = [\text{meas}(F)]^2/4.$$

This completes the proof of Lemma 1. ∎

LEMMA 2. *Let $E \subset [0, 1]$ be a Lebesgue measurable set, $\mu = \text{meas}(E) > 0$, and let h be a real number such that*

$$0 < h < \mu/n, \tag{2.3}$$

where n is a positive integer. Then there exist some points $x_j \in E$, $j = 0, 1, ..., n$, $x_0 < x_1 < \cdots < x_n$, such that

$$x_j - x_{j-1} \equiv 0 \pmod{h}, \qquad j = 1, ..., n. \tag{2.4}$$

Proof. Let $h^{-1}E := \{x = h^{-1}t : t \in E\}$, and let $\chi_{h^{-1}E}(x)$ be the characteristic function of the set $h^{-1}E$. Setting

$$g(x) = \sum_{k=0}^{\infty} \chi_{h^{-1}E}(x + k), \qquad x \in [0, 1),$$

we get

$$\int_0^1 g(x)\, dx = \text{meas}(h^{-1}E) = h^{-1}\mu > n.$$

There is, at least, one point $y_* \in [0, 1)$ where $g(y_*) \geqslant n+1$. In other words, we can find in $h^{-1}E$ some points $y_0 < y_1 < \cdots < y_n$ such that $y_j - y_{j-1} \equiv 0 \pmod 1$, $j = 1, ..., n$. The points $x_j = hy_j$, $j = 1, ..., n$, satisfy (2.4). ∎

3. Results

The main result of this paper can be stated as follows.

THEOREM 1. *Let $\{L_n\}_{n \in \mathbb{N}}$ be a sequence of positive linear operators of rank $n+1$ mapping $C[0, 1]$ into $B(\Omega)$, where $\Omega \subset [0, 1]$ is a measurable set. Suppose that $L_n(f; \cdot)$ is a measurable function for each f and each n. Furthermore, let $\gamma = \{n_i\}$, where $n_1 < n_2 < \cdots < n_i < \cdots$, and*

$$e_\gamma = \{x\colon \lim_{i \to \infty} n_i^2 \, |L_{n_i}(t^j; x) - x^j| = 0; j = 0, 1, 2; x \in \Omega\}. \tag{3.1}$$

Then $\operatorname{meas}(e_\gamma) = 0$.

Proof. It is quite simple to show that e_γ is measurable. Assume $\operatorname{meas}(e_\gamma) > 0$. Let $\delta > 0$ so that $\operatorname{meas}(e_\gamma)/2 > \delta$. Then by Egorov's theorem, there exists a measurable set $v \subset e_\gamma$, $\mu = \operatorname{meas}(v) > \operatorname{meas}(e_\gamma) - \delta$, on which the convergence of the expressions in (3.1) is uniform:

$$\lim_{i \to \infty} n_i^2 \, \| L_{n_i}(t^j; x) - x^j \|_{B(v)} = 0, \qquad j = 0, 1, 2, \tag{3.2}$$

where $\|f\|_{B(v)} = \sup_{x \in v} |f(x)|$.

Set $D_n := \{x : L_n(f; x) = 0, \ \forall f \in C[0, 1], \ x \in \Omega\}$, and $D = \limsup D_{n_i} = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} D_{n_i}$, that is to say D is the set of all points that belong to an infinite number of the sets D_{n_i}. Then $e_\gamma \cap D \neq \varnothing$ and $\operatorname{meas}(e_\gamma \cap [\bigcup_{i=k}^{\infty} D_{n_i}]) \to 0$ as $k \to \infty$. Therefore, without loss of generality, we may assume that

$$v \cap D_{n_i} = \varnothing \qquad \text{for} \quad n_i \geqslant N. \tag{3.3}$$

Fix $n_i \geqslant N$. By Lemma 2, we may find some points $x_k \in v$ $(k = 0, 1, ..., n_i + 1)$, $0 \leqslant x_0 < x_1 < \cdots < x_{n_i+1} \leqslant 1$, for which $x_k - x_{k-1} \equiv 0 \pmod h$ $(k = 1, ..., n_i + 1)$, where h is a positive number so that $h < \mu/(n_i + 1)$, $\mu = \operatorname{meas}(v)$.

Now, let $\{u_j(x)\}_{j=0, 1, ..., n_i}$, $x \in \Omega$, be a system generating the linear space $\{L_{n_i} f : f \in C[0, 1]\} \subset B(\Omega)$. Consider the matrix

$$A = \| u_j(x_k) \|_{j=0, 1, ..., n_i; k=0, 1, ..., n_i+1}.$$

If rank $A = 0$, then $L_{n_i}(f; x_k) = \sum_{j=0}^{n_i} a_j(f) \, u_j(x_k) = 0$ for every $f \in C[0, 1]$, which implies that $\{x_k\}_{k=0, 1, ..., n_i+1} \subset v \cap D_{n_i}$. This contradicts (3.3).

Therefore $\operatorname{rank} A \neq 0$. Consider a non-trivial vector $\{\gamma_k\}_{k=0,1,...,n_i+1}$ orthogonal to all the rows of the matrix A:

$$\sum_{k=0}^{n_i+1} |\gamma_k| = 1; \qquad \sum_{k=0}^{n_i+1} \gamma_k u_j(x_k) = 0, \qquad j = 0, 1, ..., n_i. \tag{3.4}$$

Now, define a continuous function $\tilde{h}$ on $[0, 1]$ by the conditions $\tilde{h}(x_k) = \operatorname{sgn} \gamma_k$, $k = 0, 1, ..., n_i + 1$; $\tilde{h}(0) = \tilde{h}(x_0)$, $\tilde{h}(1) = \tilde{h}(x_{n_i+1})$, $\tilde{h}(x)$ is linear on each interval $[0, x_0]$, $[x_0, x_1]$, ..., $[x_{n_i}, x_{n_i+1}]$, and $[x_{n_i+1}, 1]$. Then $\tilde{h} \in \mathrm{Lip}_{2/h}\, 1$ and $\|\tilde{h}\|_{C[0,1]} = 1$.

The function $L_{n_i}(\tilde{h}; x)$, $x \in \Omega$, belongs to the linear space spanned by $u_j(x)$, $x \in \Omega$. Hence from (3.4),

$$\sum_{k=0}^{n_i+1} \gamma_k L_{n_i}(\tilde{h}; x_k) = 0. \tag{3.5}$$

But then

$$\begin{aligned} 1 = \sum_{k=0}^{n_i+1} |\gamma_k| &= \sum_{k=0}^{n_i+1} \gamma_k \tilde{h}(x_k) = \sum_{k=0}^{n_i+1} \gamma_k [\tilde{h}(x_k) - L_{n_i}(\tilde{h}; x_k)] \\ &\leqslant \sum_{k=0}^{n_i+1} |\gamma_k| \, |L_{n_i}(\tilde{h}; x_k) - \tilde{h}(x_k)| \leqslant \|L_{n_i}(\tilde{h}; x) - \tilde{h}(x)\|_{B(v)}. \end{aligned} \tag{3.6}$$

On the other hand, from the Cauchy–Schwarz inequality for positive linear functionals, we obtain for $x \in \Omega$

$$\begin{aligned} &|L_{n_i}(\tilde{h}; x) - \tilde{h}(x)| \\ &\quad \leqslant L_{n_i}(|\tilde{h}(t) - \tilde{h}(x)|; x) + |\tilde{h}(x)| \, |L_{n_i}(1; x) - 1| \\ &\quad \leqslant 2h^{-1} L_{n_i}(|t - x|; x) + |L_{n_i}(1; x) - 1| \\ &\quad \leqslant 2h^{-1} [L_{n_i}((t-x)^2; x) \cdot L_{n_i}(1; x)]^{1/2} + |L_{n_i}(1; x) - 1|. \end{aligned} \tag{3.7}$$

Then by (3.6) and letting h tend to $\mu/(n_i + 1)$, we have

$$\begin{aligned} &\mu^2 [1 - \|L_{n_i}(1; x) - 1\|_{B(v)}]^2 / 4 \, \|L_{n_i}(1; x)\|_{B(v)} \\ &\quad \leqslant (n_i + 1)^2 \, \|L_{n_i}((t-x)^2; x)\|_{B(v)} \\ &\quad \leqslant (n_i + 1)^2 \, [\|L_{n_i}(1; x) - 1\|_{B(v)} \\ &\qquad + 2 \, \|L_{n_i}(t; x) - x\|_{B(v)} + \|L_{n_i}(t^2; x) - x^2\|_{B(v)}]. \end{aligned} \tag{3.8}$$

By virtue of (3.2), the left and right hand sides of (3.8) tend respectively to $\mu^2/4$ and 0 as $n_i \to \infty$, which contradicts $\mu > 0$. Thus we infer that $\operatorname{meas}(e_\gamma) = 0$. ∎

Remark. In [4], it was shown that e_γ can be non-empty and even uncountable (Theorem 2).

THEOREM 2. *Let $\mathscr{L}_n$, $n \in \mathbb{N}$, be the class of positive linear operators L_n of rank $n+1$ mapping $C[0, 1]$ into $B(\Omega)$ with $\Omega \subset [0, 1]$ and satisfying the condition $L_n(1; x) \equiv 1$. Further, let Δ be a subset of Ω and $\bar{\Delta}$ the closure of Δ. Then*

$$\inf_{L_n \in \mathscr{L}_n} \| L_n((t-x)^2; x)\|_{B(\Delta)} = [\text{meas}(\bar{\Delta})]^2/4(n+1)^2 + \alpha_n/n^2, \qquad (3.9)$$

where $\alpha_n = \alpha_n(\Delta) \geqslant 0$ for each n and $\alpha_n \to 0$ as $n \to +\infty$.

Proof. By virtue of Lemma 1, it is sufficient to prove the inequality

$$\inf_{L_n \in \mathscr{L}_n} \| L_n((t-x)^2; x)\|_{B(\Delta)} \geqslant [\text{meas}(\bar{\Delta})]^2/4(n+1)^2. \qquad (3.10)$$

Suppose that $\text{meas}(\bar{\Delta}) > 0$ and $0 < h < h' < \text{meas}(\bar{\Delta})/(n+1)$. There exist, by Lemma 2, some points $x_0^* < x_1^* < \cdots < x_{n+1}^*$, $x_k^* \in \bar{\Delta}$, with $x_k^* - x_{k-1}^* \equiv 0 \pmod{h'}$, $k = 1, 2, ..., n+1$. Then we can find $n+2$ points $x_0 < x_1 < \cdots < x_{n+1}$, $x_k \in \Delta$, such that $x_k - x_{k-1} > h$, $k = 1, 2, ..., n+1$.

Let L_n, $n \in \mathbb{N}$, be a positive linear operator of rank $n+1$ from $C[0, 1]$ into $B(\Omega)$ with $L_n(1; x) = 1$ and let $\{u_j(x)\}_{j=0, 1, ..., n}$, $x \in \Omega$, be a system generating the linear subspace $\{L_n f : f \in C[0, 1]\}$ of $B(\Omega)$. Since $1 = L_n(1; x) = \sum_{j=0}^n a_j u_j(x)$, we have $\text{rank } A = \text{rank } \|u_j(x_k)\|_{0 \leqslant j \leqslant n;\, 0 \leqslant k \leqslant n+1} \neq 0$. Now, using the arguments of the proof of Theorem 1, we obtain, by (3.7) and (3.8), the inequality

$$\| L_n((t-x)^2; x)\|_{B(\Delta)} \geqslant [\text{meas}(\bar{\Delta})]^2/4(n+1)^2,$$

which yields the inequality (3.10). ∎

As a consequence of Theorem 2 we obtain the following result.

THEOREM 3. *Let $\{L_n\}_{n \in \mathbb{N}}$ be a sequence of positive linear operators of rank $n+1$ from $C[0, 1]$ into $B(\Omega)$ with $\Omega \subset [0, 1]$. Suppose that $L_n(1; x) = 1$ for each $n \in \mathbb{N}$ and $\Delta \subset \Omega$. Then*

$$\begin{aligned} 2 \| L_n(t; x) - x \|_{B(\Delta)} &+ \| L_n(t^2; x) - x^2 \|_{B(\Delta)} \\ &\geqslant \| L_n((t-x)^2; x)\|_{B(\Delta)} \geqslant [\text{meas}(\bar{\Delta})]^2/4(n+1)^2, \end{aligned} \qquad (3.11)$$

where $\bar{\Delta}$ stands for the closure of Δ.

Let $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, be the spaces of those real-valued and measurable functions which are Lebesgue integrable to the pth power over measurable set $\Omega \subset [0, 1]$. We have:

THEOREM 4. *Under the same conditions of Theorem* 1, *suppose that the spaces* $C[0, 1]$ *and* $B(\Omega)$ *are replaced by* $L^p[0, 1]$ *and* $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, *respectively. Then for each* $v \subset \Omega$, $\operatorname{meas}(v) > 0$, *there exists a constant* $C_v > 0$ *so that*

$$\| L_n(1; x) - 1 \|_{L^p(v)} + 2 \| L_n(t; x) - x \|_{L^p(v)} + \| L_n(t^2; x) - x^2 \|_{L^p(v)} \geqslant C_v/n^2. \tag{3.12}$$

Proof. If the inequality (3.12) is not true, there exists a sequence of indexes $n_0 < n_1 < \cdots < n_i < \cdots$ such that $\lim_{i \to \infty} n_i^2 [L_{n_i}(t^j; x) - x^j] = 0$, $j = 0, 1, 2$, for almost all $x \in v$. But this contradicts the proof of Theorem 1. ∎

All results, we have obtained, are still valid in the trigonometric case. In particular, the following assertion holds.

THEOREM 1*. *Under the same conditions of Theorem* 1, *suppose that the spaces* $C[0, 1]$, $B(\Omega)$, *and the set* e_γ *are replaced by* $C_{2\pi}$, $B(\Omega_*)$, *where* $\Omega_* \subset [0, 2\pi)$ *is a measurable set, and*

$$e_\gamma^* = \{x: \lim_{i \to \infty} n_i^2 \, | L_{n_i}(1; x) - 1 | = 0,$$
$$\lim_{i \to \infty} n_i^2 \, | L_{n_i}(\cos t; x) - \cos x | = 0,$$
$$\lim_{i \to \infty} n_i^2 \, | L_{n_i}(\sin t; x) - \sin x | = 0; x \in \Omega_* \},$$

respectively. Then $\operatorname{meas}(e_\gamma^*) = 0$.

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